

TOPOSES FROM FORCING FOR INTUITIONISTIC ZF WITH ATOMS

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ABSTRACT. We introduce the forcing model of IZFA (Intuitionistic Zermelo-Fraenkel set theory with Atoms) for every Grothendieck topology and prove that the topos of sheaves on every site is equivalent to the category of ‘sets in this forcing model’.

1. INTRODUCTION

For a complete Heyting algebra H , the *Heyting-valued model* $V^{(H)}$ of *Intuitionistic Zermelo-Fraenkel set theory* (IZF) is obtained by carrying out the definition of the Boolean-valued model $V^{(B)}$ of ZFC with H in place of a complete Boolean algebra B . Then it can be shown [1, pp. 179–181] that the topos $\text{Sh}(H)$ of sheaves on H is equivalent to the category $\text{Set}^{(H)}$ of ‘sets in $V^{(H)}$ ’, which is defined more precisely as follows:

- we identify elements u, v of $V^{(H)}$ when the truth value $\|u = v\|_{V^{(H)}} \in H$ is equal to 1,
- the objects of $\text{Set}^{(H)}$ are the (identified) elements of $V^{(H)}$,
- the arrows of $\text{Set}^{(H)}$ are those (identified) elements f of $V^{(H)}$ for which $\|f \text{ is a function}\|_{V^{(H)}} = 1$.

In this paper, for every Grothendieck topology J on every small category \mathcal{C} , we introduce the *forcing model of IZFA* (*Intuitionistic Zermelo-Fraenkel set theory with Atoms*) as an extended version of Heyting-valued models of IZF and prove that the topos $\text{Sh}(\mathcal{C}, J)$ of sheaves on (\mathcal{C}, J) is equivalent to the category $\text{Set}^{(\mathcal{C}, J)}$ of ‘sets in this forcing model’.

In section 2, we define forcing for IZFA and present some propositions on it. In section 3, we define the category $\text{Set}^{(\mathcal{C}, J)}$ for each site (\mathcal{C}, J) and prove that the categories $\text{Sh}(\mathcal{C}, J)$ and $\text{Set}^{(\mathcal{C}, J)}$ are equivalent, which is the main theorem (Theorem 3.14).

Notation and terminology:

- On Grothendieck topologies or sheaves, we adopt the terminology of [5, Chapter III].
- $\text{Ob}(\mathcal{C})$ is the class of all objects of a category \mathcal{C} .
- $\text{Arr}(\mathcal{C})$ is the class of all arrows of a category \mathcal{C} .
- $\text{Hom}_{\mathcal{C}}(\text{any}, B) := \bigcup_{A \in \text{Ob}(\mathcal{C})} \text{Hom}_{\mathcal{C}}(A, B)$.
- $\text{Hom}_{\mathcal{C}}(A, \text{any}) := \bigcup_{B \in \text{Ob}(\mathcal{C})} \text{Hom}_{\mathcal{C}}(A, B)$.

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- \mathcal{L}_\in is the first-order language with two binary predicate symbols $=$ (equality), \in (membership).
- $\mathcal{L}_{\text{atom}}$ is the first-order language obtained by adding two unary predicate symbols $*$: atom, $*$: set to \mathcal{L}_\in .

2. FORCING FOR IZFA

Let (\mathcal{C}, J) be a site.

In this section, we introduce the forcing model $(W^{(\mathcal{C}, J)}, \Vdash_{(\mathcal{C}, J)})$, which consists of the class-valued presheaf $W^{(\mathcal{C}, J)}$ and the forcing relation $\Vdash_{(\mathcal{C}, J)}$. The definition of this forcing is a modification of forcing for IZF in [6]. After giving the definition, we present some propositions on it, which are used in the next section. Most proofs of these propositions are omitted in this paper since we can prove them almost by arguments similar to that of forcing for ZFC with posets familiar to set theorists.

2.1. Definition of forcing. We fix two injective class functions $x \mapsto x^{(\text{atom})}$ and $x \mapsto x^{(\text{set})}$ on V whose ranges $\{x^{(\text{atom})} \mid x \in V\}$ and $\{x^{(\text{set})} \mid x \in V\}$ are disjoint (for example, $x^{(\text{atom})} := (x, 0)$ and $x^{(\text{set})} := (x, 1)$).

Definition 2.1. We define a presheaf $W_\alpha^{(\mathcal{C}, J)}: \mathcal{C}^{\text{op}} \rightarrow \text{Set}$ for each ordinal α by transfinite recursion as follows:

[Case: $\alpha = 0$] For $A \in \text{Ob}(\mathcal{C})$,

$$W_0^{(\mathcal{C}, J)}(A) := \left\{ k^{(\text{atom})} \mid k \in \text{Hom}_{\mathcal{C}}(A, \text{any}) \right\}.$$

For $f \in \text{Hom}_{\mathcal{C}}(A, B)$, we define a function $W_0^{(\mathcal{C}, J)}(f): W_0^{(\mathcal{C}, J)}(B) \rightarrow W_0^{(\mathcal{C}, J)}(A)$ by

$$W_0^{(\mathcal{C}, J)}(f) \left(k^{(\text{atom})} \right) := (k \circ f)^{(\text{atom})}.$$

[Case: successor ordinal $\alpha + 1$] For $A \in \text{Ob}(\mathcal{C})$, we define $W_{\alpha+1}^{(\mathcal{C}, J)}(A)$ to be the set of all $a^{(\text{set})}$ satisfying two conditions (1) and (2):

- (1) $a \subseteq \bigcup_{f \in \text{Hom}_{\mathcal{C}}(\text{any}, A)} W_\alpha^{(\mathcal{C}, J)}(\text{dom } f) \times \{f\}$,
- (2) $(W_\alpha^{(\mathcal{C}, J)}(g)(b), f \circ g) \in a$ for all $(b, f) \in a$ and all $g \in \text{Hom}_{\mathcal{C}}(\text{any}, \text{dom } f)$.

For $f \in \text{Hom}_{\mathcal{C}}(A, B)$ we define a function $W_{\alpha+1}^{(\mathcal{C}, J)}(f): W_{\alpha+1}^{(\mathcal{C}, J)}(B) \rightarrow W_{\alpha+1}^{(\mathcal{C}, J)}(A)$ by

$$W_{\alpha+1}^{(\mathcal{C}, J)}(f) \left(x^{(\text{set})} \right) := \{(y, g) \mid g \in \text{Hom}_{\mathcal{C}}(\text{any}, A), (y, f \circ g) \in x\}^{(\text{set})}.$$

[Case: limit ordinal γ] For $A \in \text{Ob}(\mathcal{C})$,

$$W_\gamma^{(\mathcal{C}, J)}(A) := \bigcup_{\alpha < \gamma} W_\alpha^{(\mathcal{C}, J)}(A).$$

For $f \in \text{Hom}_{\mathcal{C}}(A, B)$, since the functions $\{W_\alpha^{(\mathcal{C}, J)}(f) \mid \alpha < \gamma\}$ are pairwise compatible by the definition, we define

$$W_\gamma^{(\mathcal{C}, J)}(f) := \bigcup_{\alpha < \gamma} W_\alpha^{(\mathcal{C}, J)}(f).$$

Definition 2.2. We define $W^{(\mathcal{C}, J)}(A) := \bigcup_{\alpha \in \text{Ord}} W_\alpha^{(\mathcal{C}, J)}(A)$ for $A \in \text{Ob}(\mathcal{C})$. We will use dotted letters $\dot{a}, \dot{b}, \dot{c} \dots$ to denote elements of $W^{(\mathcal{C}, J)}(A)$. For $\dot{a} \in W^{(\mathcal{C}, J)}(A)$ and $f \in \text{Hom}_{\mathcal{C}}(\text{any}, A)$, we define $\dot{a} \cdot f$ to be $W_\alpha^{(\mathcal{C}, J)}(f)(\dot{a})$ for some ordinal α for

which $\dot{a} \in W_\alpha^{(\mathcal{C}, J)}(A)$. This definition is independent of choice of α since the functions $\{W_\alpha^{(\mathcal{C}, J)}(f) \mid \alpha \in \text{Ord}\}$ are pairwise compatible.

Definition 2.3. Let $A \in \text{Ob}(\mathcal{C})$ and let $\dot{a} \in W^{(\mathcal{C}, J)}(A)$. \dot{a} is *atom type* if $\dot{a} = x^{(\text{atom})}$ for some x . \dot{a} is *set type* if $\dot{a} = x^{(\text{set})}$ for some x . When $\dot{a} = x^{(\text{atom})}$ or $\dot{a} = x^{(\text{set})}$, we will also write \dot{a} for x if there is no confusion.

Definition 2.4. We define the forcing relation $A \Vdash_{(\mathcal{C}, J)} \phi(\dot{a}_0, \dot{a}_1, \dots, \dot{a}_{n-1})$ for a formula $\phi(x_0, x_1, \dots, x_{n-1})$ of $\mathcal{L}_{\text{atom}}$, $A \in \text{Ob}(\mathcal{C})$, and $\dot{a}_0, \dot{a}_1, \dots, \dot{a}_{n-1} \in W^{(\mathcal{C}, J)}(A)$ as follows:

- $A \Vdash_{(\mathcal{C}, J)} \text{"}\dot{a} : \text{atom}\text{"}$ if and only if
 - (1) $\emptyset \in J(A)$ or
 - (2) \dot{a} is atom type.
- $A \Vdash_{(\mathcal{C}, J)} \text{"}\dot{a} : \text{set}\text{"}$ if and only if
 - (1) $\emptyset \in J(A)$ or
 - (2) \dot{a} is set type.
- $A \Vdash_{(\mathcal{C}, J)} \text{"}\dot{a} \in \dot{b}\text{"}$ if and only if
 - (1) $\emptyset \in J(A)$ or
 - (2)
 - (a) \dot{b} is set type and
 - (b) $\exists S \in J(A) \forall f \in S \exists \dot{x} \in W^{(\mathcal{C}, J)}(\text{dom } f)$
 - (i) $(\dot{x}, f) \in \dot{b}$ and
 - (ii) $\text{dom } f \Vdash_{(\mathcal{C}, J)} \text{"}\dot{a} \cdot f = \dot{x}\text{"}$.
- $A \Vdash_{(\mathcal{C}, J)} \text{"}\dot{a} = \dot{b}\text{"}$ if and only if
 - (1) $\emptyset \in J(A)$,
 - (2)
 - (a) \dot{a} and \dot{b} are atom type and
 - (b) $\exists S \in J(A) \forall f \in S (\dot{a} \cdot f = \dot{b} \cdot f)$, or
 - (3)
 - (a) \dot{a} and \dot{b} are set type and
 - (b) $\forall f \in \text{Hom}_{\mathcal{C}}(\text{any}, A) \forall \dot{x} \in W^{(\mathcal{C}, J)}(\text{dom } f)$
 - (i) $(\dot{x}, f) \in \dot{a} \rightarrow \text{dom } f \Vdash_{(\mathcal{C}, J)} \text{"}\dot{x} \in \dot{b} \cdot f\text{"}$ and
 - (ii) $(\dot{x}, f) \in \dot{b} \rightarrow \text{dom } f \Vdash_{(\mathcal{C}, J)} \text{"}\dot{x} \in \dot{a} \cdot f\text{"}$.
- $A \Vdash_{(\mathcal{C}, J)} \phi(\dot{a}_0, \dots, \dot{a}_{n-1})$ if and only if
 - (1) $A \Vdash_{(\mathcal{C}, J)} \phi(\dot{a}_0, \dots, \dot{a}_{n-1})$ and
 - (2) $A \Vdash_{(\mathcal{C}, J)} \psi(\dot{a}_0, \dots, \dot{a}_{n-1})$.
- $A \Vdash_{(\mathcal{C}, J)} \phi(\dot{a}_0, \dots, \dot{a}_{n-1})$ if and only if
 - (1) $A \Vdash_{(\mathcal{C}, J)} \phi(\dot{a}_0, \dots, \dot{a}_{n-1})$ or
 - (2) $A \Vdash_{(\mathcal{C}, J)} \psi(\dot{a}_0, \dots, \dot{a}_{n-1})$.
- $A \Vdash_{(\mathcal{C}, J)} \phi(\dot{a}_0, \dots, \dot{a}_{n-1})$ if and only if
 - $\forall f \in \text{Hom}_{\mathcal{C}}(\text{any}, A)$
 - (a) $\text{dom } f \Vdash_{(\mathcal{C}, J)} \phi(\dot{a}_0 \cdot f, \dots, \dot{a}_{n-1} \cdot f)$ implies
 - (b) $\text{dom } f \Vdash_{(\mathcal{C}, J)} \psi(\dot{a}_0 \cdot f, \dots, \dot{a}_{n-1} \cdot f)$.
- $A \Vdash_{(\mathcal{C}, J)} \neg \phi(\dot{a}_0, \dots, \dot{a}_{n-1})$ if and only if
 - $\forall f \in \text{Hom}_{\mathcal{C}}(\text{any}, A)$
 - (a) $\text{dom } f \Vdash_{(\mathcal{C}, J)} \phi(\dot{a}_0 \cdot f, \dots, \dot{a}_{n-1} \cdot f)$ implies
 - (b) $\emptyset \in J(\text{dom } f)$.
- $A \Vdash_{(\mathcal{C}, J)} \forall x \phi(x, \dot{a}_0, \dots, \dot{a}_{n-1})$ if and only if
 - $\forall f \in \text{Hom}_{\mathcal{C}}(\text{any}, A) \forall \dot{x} \in W^{(\mathcal{C}, J)}(\text{dom } f)$
 $\text{dom } f \Vdash_{(\mathcal{C}, J)} \phi(\dot{x}, \dot{a}_0 \cdot f, \dots, \dot{a}_{n-1} \cdot f)$.
- $A \Vdash_{(\mathcal{C}, J)} \exists x \phi(x, \dot{a}_0, \dots, \dot{a}_{n-1})$ if and only if

$$\begin{aligned} \exists S \in J(A) \forall f \in S \exists \dot{x} \in W^{(\mathcal{C}, J)}(\text{dom } f) \\ \text{dom } f \Vdash_{(\mathcal{C}, J)} \text{“} \phi(\dot{x}, \dot{a}_0 \cdot f, \dots, \dot{a}_{n-1} \cdot f) \text{”}. \end{aligned}$$

2.2. Soundness.

Definition 2.5. Let $A \in \text{Ob}(\mathcal{C})$. A sieve S on A is *J-closed* if for every $f \in \text{Hom}_{\mathcal{C}}(\text{any}, A)$, $f^*(S) \in J(\text{dom } f)$ implies $f \in S$. We define $\Omega^{(\mathcal{C}, J)}(A)$ to be the set of all *J-closed* sieves on A .

Proposition 2.6. For every $A \in \text{Ob}(\mathcal{C})$, the poset $(\Omega^{(\mathcal{C}, J)}(A), \subseteq)$ is a complete Heyting algebra in which the following properties hold:

- (1) $\bigwedge_{i \in I} S_i = \bigcap_{i \in I} S_i$,
- (2) $\bigvee_{i \in I} S_i = \{f \in \text{Hom}_{\mathcal{C}}(\text{any}, A) \mid f^*(\bigcup_{i \in I} S_i) \in J(\text{dom } f)\}$,
- (3) $S_0 \rightarrow S_1 = \{f \in \text{Hom}_{\mathcal{C}}(\text{any}, A) \mid f^*(S_0) \subseteq f^*(S_1)\}$,
- (4) $1 = \text{Hom}_{\mathcal{C}}(\text{any}, A)$,
- (5) $0 = \{f \in \text{Hom}_{\mathcal{C}}(\text{any}, A) \mid \emptyset \in J(\text{dom } f)\}$.

Proof. Straightforward. \square

Definition 2.7. Let $\phi(x_0, \dots, x_{n-1})$ be a formula of $\mathcal{L}_{\text{atom}}$ and let $A \in \text{Ob}(\mathcal{C})$. Let $\dot{a}_0, \dots, \dot{a}_{n-1} \in W^{(\mathcal{C}, J)}(A)$.

$$\begin{aligned} \|\phi(\dot{a}_0, \dots, \dot{a}_{n-1})\|_A^{(\mathcal{C}, J)} := \{f \in \text{Hom}_{\mathcal{C}}(\text{any}, A) \mid \\ \text{dom } f \Vdash_{(\mathcal{C}, J)} \text{“} \phi(\dot{a}_0 \cdot f, \dots, \dot{a}_{n-1} \cdot f) \text{”}\}. \end{aligned}$$

Proposition 2.8. Let $\phi(x_0, \dots, x_{n-1})$ be a formula of $\mathcal{L}_{\text{atom}}$ and let $A \in \text{Ob}(\mathcal{C})$. Let $\dot{a}_0, \dot{a}_1, \dots, \dot{a}_{n-1} \in W^{(\mathcal{C}, J)}(A)$. Then $\|\phi(\dot{a}_0, \dots, \dot{a}_{n-1})\|_A^{(\mathcal{C}, J)}$ is a *J-closed* sieve on A i.e. $\|\phi(\dot{a}_0, \dots, \dot{a}_{n-1})\|_A^{(\mathcal{C}, J)} \in \Omega^{(\mathcal{C}, J)}(A)$.

Proof. By induction on $\phi(x_0, \dots, x_{n-1})$. \square

Proposition 2.9. Let $\phi(x_0, \dots, x_{n-1})$ and $\psi(x_0, \dots, x_{n-1})$ be formulas of $\mathcal{L}_{\text{atom}}$. Let $A \in \text{Ob}(\mathcal{C})$ and let $\dot{a}_0, \dots, \dot{a}_{n-1} \in W^{(\mathcal{C}, J)}(A)$. Then in the complete Heyting algebra $\Omega^{(\mathcal{C}, J)}(A)$,

- (1) $\|(\phi \vee \psi)(\dot{a}_0, \dots, \dot{a}_{n-1})\|_A^{(\mathcal{C}, J)} = \|\phi(\dot{a}_0, \dots, \dot{a}_{n-1})\|_A^{(\mathcal{C}, J)} \vee \|\psi(\dot{a}_0, \dots, \dot{a}_{n-1})\|_A^{(\mathcal{C}, J)}$,
- (2) $\|(\phi \wedge \psi)(\dot{a}_0, \dots, \dot{a}_{n-1})\|_A^{(\mathcal{C}, J)} = \|\phi(\dot{a}_0, \dots, \dot{a}_{n-1})\|_A^{(\mathcal{C}, J)} \wedge \|\psi(\dot{a}_0, \dots, \dot{a}_{n-1})\|_A^{(\mathcal{C}, J)}$,
- (3) $\|(\phi \rightarrow \psi)(\dot{a}_0, \dots, \dot{a}_{n-1})\|_A^{(\mathcal{C}, J)} = \|\phi(\dot{a}_0, \dots, \dot{a}_{n-1})\|_A^{(\mathcal{C}, J)} \rightarrow \|\psi(\dot{a}_0, \dots, \dot{a}_{n-1})\|_A^{(\mathcal{C}, J)}$,
- (4) $\|\neg\phi(\dot{a}_0, \dots, \dot{a}_{n-1})\|_A^{(\mathcal{C}, J)} = \neg\|\phi(\dot{a}_0, \dots, \dot{a}_{n-1})\|_A^{(\mathcal{C}, J)}$.

Proof. Straightforward by the definition of the forcing relation. \square

Proposition 2.10. Let $\phi(x, y_0, \dots, y_{n-1})$ and $\psi(y_0, \dots, y_{n-1})$ be formulas of $\mathcal{L}_{\text{atom}}$. Let $A \in \text{Ob}(\mathcal{C})$ and let $\dot{a}_0, \dots, \dot{a}_{n-1}, \dot{b} \in W^{(\mathcal{C}, J)}(A)$.

- (1) $\|\forall x \phi(x, \dot{a}_0, \dots, \dot{a}_{n-1})\|_A^{(\mathcal{C}, J)} \leq \|\phi(\dot{b}, \dot{a}_0, \dots, \dot{a}_{n-1})\|_A^{(\mathcal{C}, J)}$,
- (2) $\|\phi(\dot{b}, \dot{a}_0, \dots, \dot{a}_{n-1})\|_A^{(\mathcal{C}, J)} \leq \|\exists x \phi(x, \dot{a}_0, \dots, \dot{a}_{n-1})\|_A^{(\mathcal{C}, J)}$,
- (3) $\|\forall x (\psi(\dot{a}_0, \dots, \dot{a}_{n-1}) \rightarrow \phi(x, \dot{a}_0, \dots, \dot{a}_{n-1}))\|_A^{(\mathcal{C}, J)} \leq \|\psi(\dot{a}_0, \dots, \dot{a}_{n-1}) \rightarrow \forall x \phi(x, \dot{a}_0, \dots, \dot{a}_{n-1})\|_A^{(\mathcal{C}, J)}$,

$$(4) \quad \|\forall x(\phi(x, \dot{a}_0, \dots, \dot{a}_{n-1}) \rightarrow \psi(\dot{a}_0, \dots, \dot{a}_{n-1}))\|_A^{(C, J)} \\ \leq \|(\exists x\phi(x, \dot{a}_0, \dots, \dot{a}_{n-1})) \rightarrow \psi(\dot{a}_0, \dots, \dot{a}_{n-1})\|_A^{(C, J)}.$$

Proof. Straightforward by the definition of the forcing relation. \square

Proposition 2.11. *Let $\phi(x, y_0, \dots, y_{n-1})$ be a formula of $\mathcal{L}_{\text{atom}}$.*

If $\|\phi(\dot{a}, \dot{b}_0, \dots, \dot{b}_{n-1})\|_A^{(C, J)} = 1$ for all $A \in \text{Ob}(\mathcal{C})$ and all $\dot{a}, \dot{b}_0, \dots, \dot{b}_{n-1} \in W^{(C, J)}(A)$, then $\|\forall x\phi(x, \dot{b}_0, \dots, \dot{b}_{n-1})\|_A^{(C, J)} = 1$ for all $A \in \text{Ob}(\mathcal{C})$ and all $\dot{b}_0, \dots, \dot{b}_{n-1} \in W^{(C, J)}(A)$.

Proof. Straightforward by the definition of the forcing relation. \square

Proposition 2.12. *Let $A \in \text{Ob}(\mathcal{C})$ and let $\dot{a}, \dot{b}, \dot{c} \in W^{(C, J)}(A)$.*

- (1) $\|\dot{a} = \dot{a}\|_A^{(C, J)} = 1$,
- (2) $\|\dot{a} = \dot{b}\|_A^{(C, J)} \leq \|\dot{b} = \dot{a}\|_A^{(C, J)}$,
- (3) $\|\dot{a} = \dot{b}\|_A^{(C, J)} \wedge \|\dot{b} = \dot{c}\|_A^{(C, J)} \leq \|\dot{a} = \dot{c}\|_A^{(C, J)}$,
- (4) $\|\dot{a} \in \dot{b}\|_A^{(C, J)} \wedge \|\dot{a} = \dot{c}\|_A^{(C, J)} \leq \|\dot{c} \in \dot{b}\|_A^{(C, J)}$,
- (5) $\|\dot{a} \in \dot{b}\|_A^{(C, J)} \wedge \|\dot{b} = \dot{c}\|_A^{(C, J)} \leq \|\dot{a} \in \dot{c}\|_A^{(C, J)}$,
- (6) $\|\dot{a} : \text{atom}\|_A^{(C, J)} \wedge \|\dot{a} = \dot{b}\|_A^{(C, J)} \leq \|\dot{b} : \text{atom}\|_A^{(C, J)}$,
- (7) $\|\dot{a} : \text{set}\|_A^{(C, J)} \wedge \|\dot{a} = \dot{b}\|_A^{(C, J)} \leq \|\dot{b} : \text{set}\|_A^{(C, J)}$.

Proof. (1): By induction on \dot{a} .

(2), (6), (7): Straightforward by the definition of the forcing relation.

(3), (4), (5): By simultaneous induction on $\dot{a}, \dot{b}, \dot{c}$. \square

Theorem 2.13. *Let $\phi(x_0, \dots, x_{n-1})$ be a formula of $\mathcal{L}_{\text{atom}}$. If ϕ is provable in intuitionistic first-order logic with equality, then $A \Vdash_{(C, J)} \phi(\dot{a}_0, \dots, \dot{a}_{n-1})$ for all $A \in \text{Ob}(\mathcal{C})$ and all $\dot{a}_0, \dots, \dot{a}_{n-1} \in W^{(C, J)}(A)$.*

Proof. It is sufficient to show that $\|\phi(\dot{a}_0, \dots, \dot{a}_{n-1})\|_A^{(C, J)} = 1$ for all $A \in \text{Ob}(\mathcal{C})$ and all $\dot{a}_0, \dots, \dot{a}_{n-1} \in W^{(C, J)}(A)$, but it is straightforward by propositions 2.9, 2.10, 2.11, and 2.12. \square

2.3. Bounded quantifiers.

Proposition 2.14. *Let $\phi(x, y, z_0, \dots, z_{n-1})$ be a formula of $\mathcal{L}_{\text{atom}}$. Let $A \in \text{Ob}(\mathcal{C})$ and let $\dot{a}, \dot{b}_0, \dots, \dot{b}_{n-1} \in W^{(C, J)}(A)$. We assume that \dot{a} is set type. Then the followings are equivalent:*

- (1) $A \Vdash_{(C, J)} \forall x \in \dot{a} \phi(x, \dot{a}, \dot{b}_0, \dots, \dot{b}_{n-1})$,
- (2) $\text{dom } f \Vdash_{(C, J)} \phi(\dot{x}, \dot{a} \cdot f, \dot{y}_0 \cdot f, \dots, \dot{y}_{n-1} \cdot f)$ for every $(\dot{x}, f) \in \dot{a}$.

Proof. Straightforward. \square

Proposition 2.15. *Let $\phi(x, y, z_0, \dots, z_{n-1})$ be a formula of $\mathcal{L}_{\text{atom}}$. Let $A \in \text{Ob}(\mathcal{C})$ and let $\dot{a}, \dot{b}_0, \dots, \dot{b}_{n-1} \in W^{(C, J)}(A)$. We assume that \dot{a} is set type. Then the followings are equivalent:*

- (1) $A \Vdash_{(C, J)} \exists x \in \dot{a} \phi(x, \dot{a}, \dot{b}_0, \dots, \dot{b}_{n-1})$,
- (2) *there exists $S \in J(A)$ with the property that for every $f \in S$, there exists $\dot{x} \in W^{(C, J)}(\text{dom } f)$ for which $(\dot{x}, f) \in \dot{a}$ and $\text{dom } f \Vdash_{(C, J)} \phi(\dot{x}, \dot{a} \cdot f, \dot{b}_0 \cdot f, \dots, \dot{b}_{n-1} \cdot f)$.*

Proof. Straightforward. \square

Definition 2.16. For a set x and an object $A \in \text{Ob}(\mathcal{C})$, we define \tilde{x}^A (or $(x)^A$) to be $\{(\tilde{y}^{\text{dom } f}, f) \mid y \in x, f \in \text{Hom}_{\mathcal{C}}(\text{any}, A)\}$ recursively.

Proposition 2.17. Let x be a set and let $A \in \text{Ob}(\mathcal{C})$. Then \tilde{x}^A is a element of $W^{(\mathcal{C}, J)}(A)$ which is set type, and $\tilde{x}^A \cdot f = \tilde{x}^{\text{dom } f}$ for all $f \in \text{Hom}_{\mathcal{C}}(\text{any}, A)$.

Proof. Straightforward. \square

Theorem 2.18. Let $\phi(x_0, \dots, x_{n-1})$ be a Δ_0 -formula of \mathcal{L}_{\in} and let $A \in \text{Ob}(\mathcal{C})$. Let a_0, \dots, a_{n-1} be sets. Then

$$\|\phi(\tilde{a}_0^A, \dots, \tilde{a}_{n-1}^A)\|_A^{(\mathcal{C}, J)} = \begin{cases} 1 & \text{if } \phi(a_0, \dots, a_{n-1}) \text{ holds (in } V), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By induction on $\phi(x_0, \dots, x_{n-1})$. \square

2.4. Matching functions.

Definition 2.19. Let $A \in \text{Ob}(\mathcal{C})$ and let S be a sieve on A . A function F on S is called a *matching function* for S if the following conditions hold:

- (1) $F(f)$ is a nonempty subset of $W^{(\mathcal{C}, J)}(\text{dom } f)$ for every $f \in S$,
- (2) For every $f \in S$ and every $g \in \text{Hom}_{\mathcal{C}}(\text{any}, \text{dom } f)$, if $\dot{a} \in F(f)$ and $\dot{b} \in F(f \circ g)$, then $\text{dom } g \Vdash_{(\mathcal{C}, J)} \dot{a} \cdot g = \dot{b}$.

Definition 2.20. Let $A \in \text{Ob}(\mathcal{C})$ and let m be a matching function for some S . Let $f \in \text{Hom}_{\mathcal{C}}(\text{any}, A)$. We define a matching function $m \cdot f$ for $f^*(S)$ by

$$(m \cdot f)(g) := m(f \cdot g) \text{ for } g \in f^*(S).$$

Definition 2.21. Let $A \in \text{Ob}(\mathcal{C})$ and let S be a sieve on A . Let F be a matching function for S . We assume that all elements of $F(f)$ are set type for every $f \in S$. Then we define $\text{ama } F := \{(\dot{x}, f \circ g) \mid f \in S, \exists \dot{a} \in F(f) ((\dot{x}, g) \in \dot{a})\} \in W^{(\mathcal{C}, J)}(A)$.

Proposition 2.22. Let $A \in \text{Ob}(\mathcal{C})$ and let S be a sieve on A . Let F be a matching function for S . We assume that all elements of $F(f)$ are set type for every $f \in S$. Let $\dot{a} := \text{ama } F$. Then $\text{dom } f \Vdash_{(\mathcal{C}, J)} \dot{a} \cdot f = \dot{b}$ for all $f \in S$ and all $\dot{b} \in F(f)$.

Proof. Straightforward by the definition of the forcing relation. \square

Theorem 2.23. Let $\phi(x, y_0, \dots, y_{n-1})$ be a formula of $\mathcal{L}_{\text{atom}}$ and let $A \in \text{Ob}(\mathcal{C})$. Let $\dot{a}_0, \dots, \dot{a}_{n-1} \in W^{(\mathcal{C}, J)}(A)$. If $A \Vdash_{(\mathcal{C}, J)} \exists! x: \text{set}, \phi(x, \dot{a}_0, \dots, \dot{a}_{n-1})$, then there exists $\dot{x} \in W^{(\mathcal{C}, J)}(A)$ for which $A \Vdash_{(\mathcal{C}, J)} \dot{x}: \text{set} \wedge \phi(\dot{x}, \dot{a}_0, \dots, \dot{a}_{n-1})$.

Proof. Straightforward by proposition 2.22. \square

2.5. IZFA.

Definition 2.24. *Intuitionistic Zermelo-Fraenkel set theory with atoms* (or IZFA) is the theory in $\mathcal{L}_{\text{atom}}$ based on the following axioms:

- (1) Set existence

$$\exists x (x: \text{set}).$$

- (2) Extensionality

$$\forall x: \text{set} \forall y: \text{set} (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y).$$

(3) Separation

$$\forall u: \text{set } \exists v: \text{set } \forall x (x \in v \leftrightarrow x \in u \wedge \phi(x)),$$

where v is not free in the formula $\phi(x)$ of $\mathcal{L}_{\text{atom}}$.

(4) Collection

$$\forall u: \text{set } (\forall x \in u \exists y \phi(x, y) \rightarrow \exists v: \text{set } \forall x \in u \exists y \in v \phi(x, y)),$$

where v is not free in the formula $\phi(x, y)$ of $\mathcal{L}_{\text{atom}}$.

(5) Pairing

$$\forall x \forall y \exists z \forall w (w \in z \leftrightarrow w = x \vee w = y).$$

(6) Union

$$\forall u: \text{set } \exists v: \text{set } \forall x (x \in v \leftrightarrow \exists y \in u (x \in y)).$$

(7) Power set

$$\forall u: \text{set } \exists v: \text{set } \forall x (x \in v \leftrightarrow \forall y \in x (y \in u)).$$

(8) Infinity

$$\exists u (\emptyset \in u \wedge \forall x \in u (x \cup \{x\} \in u)).$$

(9) \in -induction

$$\forall x (\forall y \in x \phi(y) \rightarrow \phi(x)) \rightarrow \forall x \phi(x),$$

where y is not free in the formula $\phi(x)$ of $\mathcal{L}_{\text{atom}}$.

(10) Atom

$$\forall x: \text{atom } \forall y (y \notin x),$$

$$\forall x (x: \text{atom} \vee x: \text{set}),$$

$$\forall x \neg(x: \text{atom} \wedge x: \text{set}).$$

Definition 2.25. For $A \in \text{Ob}(\mathcal{C})$ and $\dot{a}, \dot{b} \in W^{(\mathcal{C}, J)}(A)$, we define $\text{up}_A(\dot{a}, \dot{b})$, $\text{op}_A(\dot{a}, \dot{b}) \in W^{(\mathcal{C}, J)}(A)$ as follows:

$$\begin{aligned} \text{up}_A(\dot{a}, \dot{b}) := & \left\{ \left(\dot{a} \cdot f, f \right) \mid f \in \text{Hom}_{\mathcal{C}}(\text{any}, A) \right\} \\ & \cup \left\{ \left(\dot{b} \cdot f, f \right) \mid f \in \text{Hom}_{\mathcal{C}}(\text{any}, A) \right\}, \end{aligned}$$

$$\text{op}_A(\dot{a}, \dot{b}) := \text{up}_A \left(\text{up}_A(\dot{a}, \dot{a}), \text{up}_A(\dot{a}, \dot{b}) \right).$$

Theorem 2.26. $A \Vdash_{(\mathcal{C}, J)} \text{“}\phi\text{”}$ holds for all axioms ϕ of IZFA and all $A \in \text{Ob}(\mathcal{C})$.

Proof. Easy. For example, $\dot{z} := \text{up}_A(\dot{x}, \dot{y})$ is a witness for (5) Pairing. \square

3. TOPOSES FROM FORCING

Let (\mathcal{C}, J) be a site. In this section, we define the category $\text{Set}^{(\mathcal{C}, J)}$ of ‘sets in the forcing model $(W^{(\mathcal{C}, J)}, \Vdash_{(\mathcal{C}, J)})$ ’ and prove the main theorem that the categories $\text{Sh}(\mathcal{C}, J)$ and $\text{Set}^{(\mathcal{C}, J)}$ are equivalent by constructing two functors $K: \text{Sh}(\mathcal{C}, J) \rightarrow \text{Set}^{(\mathcal{C}, J)}$ and $L: \text{Set}^{(\mathcal{C}, J)} \rightarrow \text{Sh}(\mathcal{C}, J)$ concretely of which the pair (K, L) is an equivalence of these categories.

3.1. Category $\text{Set}^{(\mathcal{C}, J)}$ of ‘sets in $(W^{(\mathcal{C}, J)}, \Vdash_{(\mathcal{C}, J)})$ ’.

Definition 3.1. Let $A \in \text{Ob}(\mathcal{C})$. We define an equivalence relation \sim_A on $W^{(\mathcal{C}, J)}(A)$ by

$$\dot{a} \sim_A \dot{b} \Leftrightarrow A \Vdash_{(\mathcal{C}, J)} “\dot{a} = \dot{b}”.$$

Then the quotient $W^{(\mathcal{C}, J)}(A) / \sim_A = \{[\dot{a}]_A \mid \dot{a} \in W^{(\mathcal{C}, J)}(A)\}$ will be denoted by $W_{\sim}^{(\mathcal{C}, J)}(A)$.

Definition 3.2. • A (\mathcal{C}, J) -sequence is a sequence $(x_A)_{A \in \text{Ob}(\mathcal{C})}$ of which each x_A is an element of $W_{\sim}^{(\mathcal{C}, J)}(A)$.
 • A (\mathcal{C}, J) -sequence $(x_A)_{A \in \text{Ob}(\mathcal{C})}$ is called *stable* if $\text{dom } f \Vdash_{(\mathcal{C}, J)} “\dot{a} \cdot f = \dot{b}”$ for every $f \in \text{Arr}(\mathcal{C})$, every $\dot{a} \in x_{\text{cod } f}$ and every $\dot{b} \in x_{\text{dom } f}$.
 • A (\mathcal{C}, J) -set is a stable (\mathcal{C}, J) -sequence $(x_A)_{A \in \text{Ob}(\mathcal{C})}$ for which $A \Vdash_{(\mathcal{C}, J)} “\dot{a}: \text{set}”$ for every $A \in \text{Ob}(\mathcal{C})$ and every $\dot{a} \in x_A$.

Definition 3.3. We define a category $\text{Set}^{(\mathcal{C}, J)}$ as follows:

- the objects of $\text{Set}^{(\mathcal{C}, J)}$ are the (\mathcal{C}, J) -sets,
- the arrows of $\text{Set}^{(\mathcal{C}, J)}$ from $(x_A)_{A \in \text{Ob}(\mathcal{C})}$ to $(y_A)_{A \in \text{Ob}(\mathcal{C})}$ are those (\mathcal{C}, J) -sets $(p_A)_{A \in \text{Ob}(\mathcal{C})}$ for which $A \Vdash_{(\mathcal{C}, J)} “\dot{f} \text{ is a function from } \dot{a} \text{ to } \dot{b}”$ for every $A \in \text{Ob}(\mathcal{C})$, every $\dot{a} \in x_A$, every $\dot{b} \in y_A$, and every $\dot{f} \in p_A$,
- the composition of two arrows $(p_A)_{A \in \text{Ob}(\mathcal{C})}$ and $(q_A)_{A \in \text{Ob}(\mathcal{C})}$ of $\text{Set}^{(\mathcal{C}, J)}$ is the unique arrow $(r_A)_{A \in \text{Ob}(\mathcal{C})}$ for which $A \Vdash_{(\mathcal{C}, J)} “\dot{f} \circ \dot{g} = \dot{h}”$ for every $A \in \text{Ob}(\mathcal{C})$, every $\dot{f} \in p_A$, every $\dot{g} \in q_A$, and every $\dot{h} \in r_A$.

3.2. Functor $K: \text{Sh}(\mathcal{C}, J) \rightarrow \text{Set}^{(\mathcal{C}, J)}$.

Definition 3.4. For a sheaf F on (\mathcal{C}, J) , $A \in \text{Ob}(\mathcal{C})$ and $a \in F(A)$, we define

$$\begin{aligned} \overline{a}^{F, A} := \left\{ \left(\text{op}_{\text{dom } g} \left(\check{x}^{\text{dom } g}, f^{(\text{atom})} \right), g \right) \mid f \in \text{Arr}(\mathcal{C}), x \in F(\text{cod } f), \right. \\ \left. g \in \text{Hom}_{\mathcal{C}}(\text{dom } f, A), F(f)(x) = F(g)(a) \right\}. \end{aligned}$$

Proposition 3.5. Let F be a sheaf on (\mathcal{C}, J) and let $A \in \text{Ob}(\mathcal{C})$.

- (1) $\overline{a}^{F, A} \in W^{(\mathcal{C}, J)}(A)$ for all $a \in F(A)$.
- (2) $\overline{a}^{F, A} \cdot h = \overline{F(h)(a)}^{F, \text{dom } h}$ for all $a \in F(A)$ and all $h \in \text{Hom}_{\mathcal{C}}(\text{any}, A)$.
- (3) For $a, b \in F(A)$, if $A \Vdash_{(\mathcal{C}, J)} “\overline{a}^{F, A} = \overline{b}^{F, A}”$, then $a = b$.

Proof. (1), (2): Immediate.

(3): Straightforward since $A \Vdash_{(\mathcal{C}, J)} “\left(\check{a}^A, 1_A^{(\text{atom})} \right) \in \overline{a}^{F, A}”$. □

Definition 3.6. We define a functor $K: \text{Sh}(\mathcal{C}, J) \rightarrow \text{Set}^{(\mathcal{C}, J)}$ as follows:

- $K(F) := ([\dot{K}_{F, A}]_A)_{A \in \text{Ob}(\mathcal{C})}$ for a sheaf F on (\mathcal{C}, J) ,
 where $\dot{K}_{F, A} := \{(\overline{a}^{F, \text{dom } f}, f) \mid f \in \text{Hom}_{\mathcal{C}}(\text{any}, A), a \in F(\text{dom } f)\}$,
- $K(\sigma) := ([\dot{K}_{\sigma, A}]_A)_{A \in \text{Ob}(\mathcal{C})}$ for $\sigma = (\sigma_A)_{A \in \text{Ob}(\mathcal{C})} \in \text{Hom}_{\text{Sh}(\mathcal{C}, J)}(F, G)$,

$$\begin{aligned} \text{where } \dot{K}_{\sigma, A} := \left\{ \left(\text{op}_{\text{dom } f} \left(\overline{a}^{F, \text{dom } f}, \overline{\sigma_{\text{dom } f}(a)}^{G, \text{dom } f} \right), f \right) \mid \right. \\ \left. f \in \text{Hom}_{\mathcal{C}}(\text{any}, A), a \in F(\text{dom } f) \right\}. \end{aligned}$$

3.3. Functor $L: \text{Set}^{(\mathcal{C}, J)} \rightarrow \text{Sh}(\mathcal{C}, J)$.

Definition 3.7. Let $\mathbf{a} = ([\dot{a}_A]_A)_{A \in \text{Ob}(\mathcal{C})} \in \text{Ob}(\text{Set}^{(\mathcal{C}, J)})$ and let $A \in \text{Ob}(\mathcal{C})$.

- We define $M_{\mathbf{a}, A}$ to be the set of all matching functions m with the two properties that $\text{dom}(m) \in J(A)$ and that for every $f \in \text{dom}(m)$, there exists $\dot{x} \in W^{(\mathcal{C}, J)}(\text{dom } f)$ for which $m(f) = [\dot{x}]_{\text{dom } f}$ and $\text{dom } f \Vdash “\dot{x} \in \dot{a}_{\text{dom } f}”$.
- An equivalence relation \sim_A on $M_{\mathbf{a}, A}$ is defined as follows:

$$m \sim_A m' \Leftrightarrow \text{there exists } T \in J(A) \text{ for which } T \subseteq \text{dom}(m) \cap \text{dom}(m') \\ \text{and } m(f) = m'(f) \text{ for every } f \in T.$$

Definition 3.8. For $\mathbf{a} \in \text{Ob}(\text{Set}^{(\mathcal{C}, J)})$, we define a presheaf $L_{\mathbf{a}}$ on \mathcal{C} as follows:

- for $A \in \text{Ob}(\mathcal{C})$, $L_{\mathbf{a}}(A) := M_{\mathbf{a}, A} / \sim_A = \{[m]_{\mathbf{a}, A} \mid m \in M_{\mathbf{a}, A}\}$,
- for $f \in \text{Hom}_{\mathcal{C}}(A, B)$, $L_{\mathbf{a}}(f) : L_{\mathbf{a}}(B) \rightarrow L_{\mathbf{a}}(A)$ is defined by

$$L_{\mathbf{a}}(f)([m]_{\mathbf{a}, B}) := [m \cdot f]_{\mathbf{a}, A}.$$

Proposition 3.9. $L_{\mathbf{a}}$ is a sheaf on (\mathcal{C}, J) for every $\mathbf{a} \in \text{Ob}(\text{Set}^{(\mathcal{C}, J)})$.

Proof. Similar to [5, p. 132, Lemma 5]. \square

Definition 3.10. Let $\mathbf{p} \in \text{Hom}_{\text{Set}^{(\mathcal{C}, J)}}(\mathbf{a}, \mathbf{b})$ for $\mathbf{a}, \mathbf{b} \in \text{Ob}(\text{Set}^{(\mathcal{C}, J)})$. Let $A \in \text{Ob}(\mathcal{C})$ and let $m \in M_{\mathbf{a}, A}$. Let $\mathbf{b} = ([\dot{b}_E]_E)_{E \in \text{Ob}(\mathcal{C})}$ and let $\mathbf{p} = ([\dot{p}_E]_E)_{E \in \text{Ob}(\mathcal{C})}$. We define $\text{app}_A(\mathbf{p}, m) \in M_{\mathbf{b}, A}$ as follows:

- $\text{dom}(\text{app}_A(\mathbf{p}, m)) := \{f \in \text{dom}(m) \mid \exists \dot{y} \in W^{(\mathcal{C}, J)}(\text{dom } f) \text{ } \phi(\dot{y}, f, \mathbf{p})\}$, where $\phi(\dot{y}, f, \mathbf{p})$ is the condition that $\text{dom } f \Vdash_{(\mathcal{C}, J)} “\dot{y} \in \dot{b}_{\text{dom } f}”$ and $\text{dom } f \Vdash_{(\mathcal{C}, J)} “\dot{p}_{\text{dom } f}(\dot{x}) = \dot{y}”$ for $[\dot{x}]_{\text{dom } f} := m(f)$,
- $\text{app}_A(\mathbf{p}, m)(f)$ is the unique $[\dot{y}]_{\text{dom } f} \in W_{=}^{(\mathcal{C}, J)}(\text{dom } f)$ for which $\phi(\dot{y}, f, \mathbf{p})$ holds.

Definition 3.11. Let $\mathbf{p} \in \text{Hom}_{\text{Set}^{(\mathcal{C}, J)}}(\mathbf{a}, \mathbf{b})$ and let $A \in \text{Ob}(\mathcal{C})$. A function $L_{\mathbf{p}, A} : L_{\mathbf{a}}(A) \rightarrow L_{\mathbf{b}}(A)$ is defined by

$$L_{\mathbf{p}, A}([m]_{\mathbf{a}, A}) := [\text{app}_A(\mathbf{p}, m)]_{\mathbf{b}, A}.$$

Definition 3.12. We define a functor $L: \text{Set}^{(\mathcal{C}, J)} \rightarrow \text{Sh}(\mathcal{C}, J)$ as follows:

- $L(\mathbf{a}) := L_{\mathbf{a}}$ for $\mathbf{a} \in \text{Ob}(\text{Set}^{(\mathcal{C}, J)})$,
- for $\mathbf{p} \in \text{Hom}_{\text{Set}^{(\mathcal{C}, J)}}(\mathbf{a}, \mathbf{b})$, a natural transformation $L(\mathbf{p}): L(\mathbf{a}) \rightarrow L(\mathbf{b})$ is defined by $L(\mathbf{p}) := (L_{\mathbf{p}, A})_{A \in \text{Ob}(\mathcal{C})}$.

3.4. Main theorem.

Definition 3.13. Let $A \in \text{Ob}(\mathcal{C})$ and let $\dot{x} \in W^{(\mathcal{C}, J)}(A)$. We define a matching function $m_{\dot{x}}$ for $\text{Hom}_{\mathcal{C}}(\text{any}, A)$ by $m_{\dot{x}}(f) := [\dot{x} \cdot f]_{\text{dom } f}$.

Now we will prove the main theorem:

Theorem 3.14. (1) $K \circ L \cong 1_{\text{Set}^{(\mathcal{C}, J)}}$.
 (2) $L \circ K \cong 1_{\text{Sh}(\mathcal{C}, J)}$.

Thus $\text{Sh}(\mathcal{C}, J)$ and $\text{Set}^{(\mathcal{C}, J)}$ are equivalent.

Proof of (1). For each $\mathbf{a} \in \text{Ob}(\text{Set}^{(\mathcal{C}, J)})$ and each $A \in \text{Ob}(\mathcal{C})$, we define

$$\dot{P}_{\mathbf{a}, A} := \left\{ \left(\text{op}_{\text{dom } f} \left(\dot{x}, \overline{[m_{\dot{x}}]_{\mathbf{a}, \text{dom } f}}^{L_{\mathbf{a}, \text{dom } f}} \right), f \right) \mid \exists \dot{y} \in \mathbf{a}(A) \text{ } ((\dot{x}, f) \in \dot{y}) \right\}.$$

Let $\mathbf{P}_{\mathbf{a}} := ([\dot{P}_{\mathbf{a},A}]_A)_{A \in \text{Ob}(\mathcal{C})}$ for each $\mathbf{a} \in \text{Ob}(\text{Set}^{(\mathcal{C},J)})$.

We will prove that $(\mathbf{P}_{\mathbf{a}})_{\mathbf{a} \in \text{Ob}(\text{Set}^{(\mathcal{C},J)})}$ is a natural isomorphism from $1_{\text{Set}^{(\mathcal{C},J)}}$ to $K \circ L$. Since $K \circ L(\mathbf{a}) = ([\dot{K}_{L\mathbf{a},A}]_A)_{A \in \text{Ob}(\mathcal{C})}$ and $K \circ L(\mathbf{Q}) = ([\dot{K}_{L(\mathbf{Q}),A}]_A)_{A \in \text{Ob}(\mathcal{C})}$ for every $\mathbf{a} \in \text{Ob}(\mathcal{C})$ and every $\mathbf{Q} \in \text{Arr}(\text{Set}^{(\mathcal{C},J)})$, it is sufficient to show the following:

- (a) $A \Vdash_{(\mathcal{C},J)}$ “ $\dot{P}_{\mathbf{a},A}$ is a bijection from \dot{a}_A to $\dot{K}_{L\mathbf{a},A}$ ” for every $A \in \text{Ob}(\mathcal{C})$ and every $\mathbf{a} = ([\dot{a}_E]_E)_{E \in \text{Ob}(\mathcal{C})} \in \text{Ob}(\text{Set}^{(\mathcal{C},J)})$,
- (b) For every $\mathbf{Q} = ([\dot{Q}_E]_E)_{E \in \text{Ob}(\mathcal{C})} \in \text{Hom}_{\text{Set}^{(\mathcal{C},J)}}(\mathbf{a}, \mathbf{b})$, the diagram (1) commutes,

$$\begin{array}{ccc} \mathbf{a} & \xrightarrow{\mathbf{P}_{\mathbf{a}}} & K \circ L(\mathbf{a}) \\ \mathbf{Q} \downarrow & & \downarrow K \circ L(\mathbf{Q}) \\ \mathbf{b} & \xrightarrow{\mathbf{P}_{\mathbf{b}}} & K \circ L(\mathbf{b}) \end{array} \quad (1)$$

i.e. $A \Vdash_{(\mathcal{C},J)}$ “ $\dot{P}_{\mathbf{b},A} \circ \dot{Q}_A = \dot{K}_{L(\mathbf{Q}),A} \circ \dot{P}_{\mathbf{a},A}$ ” for every $A \in \text{Ob}(\mathcal{C})$.

For (a): It is easy to prove that $A \Vdash_{(\mathcal{C},J)}$ “ $\dot{P}_{\mathbf{a},A}$ is a function from \dot{a}_A to $\dot{K}_{L\mathbf{a},A}$ ”. First, we will show that $A \Vdash_{(\mathcal{C},J)}$ “ $\dot{P}_{\mathbf{a},A}$ is a injection from \dot{a}_A to $\dot{K}_{L\mathbf{a},A}$ ”. By proposition 2.14, it is sufficient to show that for every $f \in \text{Hom}_{\mathcal{C}}(\text{any}, A)$ and every $\dot{x}, \dot{x}' \in W^{(\mathcal{C},J)}(\text{dom } f)$, if (\dot{x}, f) and (\dot{x}', f) are in \dot{a}_A , and if $\text{dom } f \Vdash_{(\mathcal{C},J)}$ “ $\dot{P}_{\mathbf{a}, \text{dom } f}(\dot{x}) = \dot{P}_{\mathbf{a}, \text{dom } f}(\dot{x}')$ ”, then $\text{dom } f \Vdash_{(\mathcal{C},J)}$ “ $\dot{x} = \dot{x}'$ ”. Fix \dot{x}, \dot{x}' and f as above. Then

$$\begin{aligned} \text{dom } f \Vdash_{(\mathcal{C},J)} \text{ “} \overline{[m_{\dot{x}}]_{\mathbf{a}, \text{dom } f}}^{L_{\mathbf{a}, \text{dom } f}} &= \dot{P}_{\mathbf{a}, \text{dom } f}(\dot{x}) \\ &= \dot{P}_{\mathbf{a}, \text{dom } f}(\dot{x}') \\ &= \overline{[m_{\dot{x}'}]_{\mathbf{a}, \text{dom } f}}^{L_{\mathbf{a}, \text{dom } f}} \text{”}. \end{aligned}$$

By proposition 3.5, $[m_{\dot{x}}]_{\mathbf{a}, \text{dom } f} = [m_{\dot{x}'}]_{\mathbf{a}, \text{dom } f}$. Hence, there exists $S \in J(\text{dom } f)$ for which $[\dot{x} \cdot g]_{\text{dom } g} = [\dot{x}' \cdot g]_{\text{dom } g}$ for all $g \in S$, and $\text{dom } f \Vdash_{(\mathcal{C},J)}$ “ $\dot{x} = \dot{x}'$ ” as required.

Next, we will prove $A \Vdash_{(\mathcal{C},J)}$ “ $\dot{P}_{\mathbf{a},A}$ is a surjection from \dot{a}_A to $\dot{K}_{L\mathbf{a},A}$ ”. Fix $(\dot{y}, f) \in \dot{K}_{L\mathbf{a},A}$. By proposition 2.14 and 2.15, it is sufficient to show that there exists $S \in J(\text{dom } f)$ with the property that for every $g \in S$, there exists $\dot{x} \in W^{(\mathcal{C},J)}(\text{dom } g)$ for which $(\dot{x}, g) \in \dot{a}_{\text{dom } f}$ and $\text{dom } g \Vdash_{(\mathcal{C},J)}$ “ $\dot{y} \cdot g = \dot{P}_{\mathbf{a}, \text{dom } g}(\dot{x})$ ”. Now we have $m \in M_{\mathbf{a}, \text{dom } f}$ for which $\dot{y} = \overline{[m]_{\mathbf{a}, \text{dom } f}}^{L_{\mathbf{a}, \text{dom } f}}$ since $(\dot{y}, f) \in \dot{K}_{L\mathbf{a},A}$. By the definition of $M_{\mathbf{a}, \text{dom } f}$, for every $h \in \text{dom}(m)$, there exists $\dot{z}_h \in W^{(\mathcal{C},J)}(\text{dom } h)$ for which $m(h) = [\dot{z}_h]_{\text{dom } h}$ and $\text{dom } h \Vdash_{(\mathcal{C},J)}$ “ $\dot{z}_h \in \dot{a}_{\text{dom } h} = \dot{a}_{\text{dom } f} \cdot h$ ”. Then there exists $T_h \in J(\text{dom } h)$ with the property that for every $k \in T_h$, there exists $\dot{x} \in W^{(\mathcal{C},J)}(\text{dom } k)$ for which $(\dot{x}, k) \in \dot{a}_{\text{dom } f} \cdot h$ and $\text{dom } k \Vdash_{(\mathcal{C},J)}$ “ $\dot{z}_h \cdot k = \dot{x}$ ”. Now we claim $S := \{h \circ k \mid h \in \text{dom}(m), k \in T_h\}$ is as required. Indeed, for every $g = h \circ k$ with $h \in \text{dom}(m)$ and with $k \in T_h$, since $m \cdot g(l) = m(g \circ l) = m(h \circ k \circ l) =$

$[z_h \cdot (k \circ l)]_{\text{dom } l} = [\dot{x} \cdot l]_{\text{dom } l} = m_{\dot{x}}(l)$ for every $l \in \text{Hom}_{\mathcal{C}}(\text{any}, \text{dom } g)$, we have

$$\begin{aligned} \text{dom } g \Vdash_{(\mathcal{C}, J)} \dot{y} \cdot g &= \overline{[m]_{\mathbf{a}, \text{dom } f}}^{L_{\mathbf{a}, \text{dom } f}} \cdot g \\ &= \overline{[m \cdot g]_{\mathbf{a}, \text{dom } g}}^{L_{\mathbf{a}, \text{dom } g}} \\ &= \overline{[m_{\dot{x}}]_{\mathbf{a}, \text{dom } g}}^{L_{\mathbf{a}, \text{dom } g}} \\ &= \dot{P}_{\mathbf{a}, \text{dom } g}(\dot{x}). \end{aligned}$$

For (b): Let $\mathbf{a} = ([\dot{a}_E]_E)_{E \in \text{Ob}(\mathcal{C})}$ and let $\mathbf{b} = ([\dot{b}_E]_E)_{E \in \text{Ob}(\mathcal{C})}$. Fix $(\dot{x}, f) \in \dot{a}_A$. By proposition 2.14, it is sufficient to prove that $\text{dom } f \Vdash_{(\mathcal{C}, J)} \dot{P}_{\mathbf{b}, \text{dom } f} \circ \dot{Q}_{\text{dom } f}(\dot{x}) = \dot{K}_{L(\mathbf{Q}), \text{dom } f} \circ \dot{P}_{\mathbf{a}, \text{dom } f}(\dot{x})$. Since $\text{dom } f \Vdash_{(\mathcal{C}, J)} \exists y \in \dot{b}_{\text{dom } f} (y = \dot{Q}_{\text{dom } f}(\dot{x}))$, there exists $S \in J(\text{dom } f)$ with the property that for every $g \in S$, there exists $\dot{y} \in W^{(\mathcal{C}, J)}(\text{dom } g)$ for which $(\dot{y}, g) \in \dot{b}_{\text{dom } f}$ and $\text{dom } g \Vdash \dot{y} = \dot{Q}_{\text{dom } g}(\dot{x} \cdot g)$. Fix $g \in S$ and fix \dot{y} as above. Then

$$\text{dom } g \Vdash_{(\mathcal{C}, J)} \dot{P}_{\mathbf{b}, \text{dom } g} \circ \dot{Q}_{\text{dom } g}(\dot{x} \cdot g) = \dot{P}_{\mathbf{b}, \text{dom } g}(\dot{y}) = \overline{[m_{\dot{y}}]_{\mathbf{b}, \text{dom } g}}^{L_{\mathbf{b}, \text{dom } g}},$$

and

$$\begin{aligned} \text{dom } g \Vdash_{(\mathcal{C}, J)} \dot{K}_{L(\mathbf{Q}), \text{dom } g} \circ \dot{P}_{\mathbf{a}, \text{dom } g}(\dot{x} \cdot g) &= \dot{K}_{L(\mathbf{Q}), \text{dom } g}(\overline{[m_{\dot{x} \cdot g}]_{\mathbf{a}, \text{dom } g}}^{L_{\mathbf{a}, \text{dom } g}}) \\ &= \overline{L_{\mathbf{Q}, \text{dom } g}([m_{\dot{x} \cdot g}]_{\mathbf{a}, \text{dom } g})}^{L_{\mathbf{b}, \text{dom } g}} \\ &= \overline{[\text{app}_{\text{dom } g}(\mathbf{Q}, m_{\dot{x} \cdot g})]_{\mathbf{b}, \text{dom } g}}^{L_{\mathbf{a}, \text{dom } g}}. \end{aligned}$$

For every $h \in \text{dom}(\text{app}_{\text{dom } g}(\mathbf{Q}, m_{\dot{x} \cdot g}))$, since $\text{dom } h \Vdash_{(\mathcal{C}, J)} \dot{y} \cdot h = \dot{Q}_{\text{dom } h}(\dot{x} \cdot (g \circ h))$, we have $\text{app}_{\text{dom } g}(\mathbf{Q}, m_{\dot{x} \cdot g})(h) = [\dot{y} \cdot h]_{\text{dom } h} = m_{\dot{y}}(h)$. Hence, $\text{dom } g \Vdash_{(\mathcal{C}, J)} \dot{P}_{\mathbf{b}, \text{dom } g} \circ \dot{Q}_{\text{dom } g}(\dot{x} \cdot g) = \dot{K}_{L(\mathbf{Q}), \text{dom } g} \circ \dot{P}_{\mathbf{a}, \text{dom } g}(\dot{x} \cdot g)$ for all $g \in S$, which proves (b). \square

Proof of (2). For a sheaf F on (\mathcal{C}, J) and $A \in \text{Ob}(\mathcal{C})$, we define a function

$$\sigma_{F, A}: F(A) \rightarrow L_{K(F)}(A)$$

by $\sigma_{F, A}(a) := [m_{\overline{a}^{F, A}}]_{K(F), A}$.

We will prove that $\sigma_F := (\sigma_{F, A})_{A \in \text{Ob}(\mathcal{C})}$ is a natural isomorphism from F to $L \circ K(F)$ for every sheaf F on (\mathcal{C}, J) and that $(\sigma_F)_{F \in \text{Ob}(\text{Sh}(\mathcal{C}, J))}$ is a natural isomorphism from $1_{\text{Sh}(\mathcal{C}, J)}$ to $L \circ K$. It is sufficient to show the following:

- (a) $\sigma_{F, A}$ is a bijection from $F(A)$ to $L_{K(F)}(A)$ for every sheaf F on (\mathcal{C}, J) and for every $A \in \text{Ob}(\mathcal{C})$,
- (b) the diagram (2) commutes for every sheaf F on (\mathcal{C}, J) and for every $f \in \text{Hom}_{\mathcal{C}}(A, B)$,

$$\begin{array}{ccc} F(B) & \xrightarrow{\sigma_{F, B}} & L_{K(F)}(B) \\ F(f) \downarrow & & \downarrow L_{K(F)}(f) \\ F(A) & \xrightarrow{\sigma_{F, A}} & L_{K(F)}(A) \end{array} \quad (2)$$

- (c) the diagram (3) commutes for every $\rho = (\rho_E)_{E \in \text{Ob}(\mathcal{C})} \in \text{Hom}_{\text{Sh}(\mathcal{C}, J)}(F, G)$ and for every $A \in \text{Ob}(\mathcal{C})$.

$$\begin{array}{ccc}
 F(A) & \xrightarrow{\sigma_{F,A}} & L_{K(F)}(A) \\
 \rho_A \downarrow & & \downarrow L_{K(\rho), A} \\
 G(A) & \xrightarrow{\sigma_{G,A}} & L_{K(G)}(A)
 \end{array} \tag{3}$$

For (a): First, we will prove that $\sigma_{F,A}$ is an injection from $F(A)$ to $L_{K(F)}(A)$. Fix $a, b \in F(A)$ for which $\sigma_{F,A}(a) = \sigma_{F,A}(b)$ i.e. $[m_{\bar{a}^{F,A}}]_{K(A), A} = [m_{\bar{b}^{F,A}}]_{K(A), A}$. Then there exists $S \in J(A)$ for which $S \subseteq \text{dom}(m_{\bar{a}^{F,A}}) \cap \text{dom}(m_{\bar{b}^{F,A}})$ and $m_{\bar{a}^{F,A}}(f) = m_{\bar{b}^{F,A}}(f)$ for every $f \in S$. By Definition 3.13, $\text{dom } f \Vdash_{(\mathcal{C}, J)} \text{“}\bar{a}^{F,A} \cdot f = \bar{b}^{F,A} \cdot f\text{”}$ for every $f \in S$. By Proposition 2.8, $A \Vdash_{(\mathcal{C}, J)} \text{“}\bar{a}^{F,A} = \bar{b}^{F,A}\text{”}$. Hence, by Proposition 3.5 (3), $a = b$.

Next, we will prove that $\sigma_{F,A}$ is a surjection from $F(A)$ to $L_{K(F)}(A)$. Fix $[m]_{K(F), A} \in L_{K(F)}(A)$. Since $m \in M_{K(F), A}$, for each $f \in \text{dom}(m)$ there exists $\dot{x}_f \in W^{(\mathcal{C}, J)}(\text{dom } f)$ for which $m(f) = [\dot{x}_f]_{\text{dom } f}$ and $\text{dom } f \Vdash_{(\mathcal{C}, J)} \text{“}\dot{x}_f \in \dot{K}_{F, \text{dom } f}\text{”}$. By the definition of $\dot{K}_{F, \text{dom } f}$, there exists $S_f \in J(\text{dom } f)$ with the property that for every $g \in S_f$, there exists $a_{f,g} \in F(\text{dom } g)$ for which $\text{dom } g \Vdash_{(\mathcal{C}, J)} \text{“}\dot{x}_f \cdot g = \bar{a}_{f,g}^{F, \text{dom } g}\text{”}$. Then for every $f \in \text{dom}(m)$, we claim that $(a_{f,g})_{g \in S_f}$ is a matching family. Indeed, by Proposition 2.8 and Proposition 3.5 (2), for every $g \in S_f$ and every $h \in \text{Hom}_{\mathcal{C}}(\text{any}, \text{dom } g)$,

$$\begin{aligned}
 \text{dom } h \Vdash_{(\mathcal{C}, J)} \text{“}\dot{x}_f \cdot (g \circ h) &= (\dot{x}_f \cdot g) \cdot h \\
 &= \bar{a}_{f,g}^{F, \text{dom } g} \cdot h \\
 &= \overline{F(h)(a_{f,g})}^{F, \text{dom } h} \text{”}.
 \end{aligned}$$

On the other hand, since $g \circ h \in S_f$, $\text{dom } h \Vdash_{(\mathcal{C}, J)} \text{“}\dot{x}_f \cdot (g \circ h) = \bar{a}_{f, g \circ h}^{F, \text{dom } h}\text{”}$. Thus, $\text{dom } h \Vdash_{(\mathcal{C}, J)} \text{“}\overline{F(h)(a_{f,g})}^{F, \text{dom } h} = \bar{a}_{f, g \circ h}^{F, \text{dom } h}\text{”}$. By Proposition 3.5 (3), $F(h)(a_{f,g}) = a_{f, g \circ h}$, which proves the claim. Since F is a sheaf on (\mathcal{C}, J) , we obtain the amalgamation $a_f \in F(\text{dom } f)$ of $(a_{f,g})_{g \in S_f}$ for each $f \in \text{dom}(m)$. Then $\text{dom } f \Vdash_{(\mathcal{C}, J)} \text{“}\dot{x}_f = \bar{a}_f^{F, \text{dom } f}\text{”}$ for every $f \in \text{dom}(m)$ since

$$\begin{aligned}
 \text{dom } g \Vdash_{(\mathcal{C}, J)} \text{“}\dot{x}_f \cdot g &= \bar{a}_{f,g}^{F, \text{dom } g} \\
 &= \overline{F(g)(a_f)}^{F, \text{dom } g} \\
 &= \bar{a}_f^{F, \text{dom } f} \cdot g \text{”}
 \end{aligned}$$

for every $g \in S_f$. So we see that $(a_f)_{f \in \text{dom}(m)}$ is also a matching family since m is a matching function such that $m(f) = [\dot{x}_f]_{\text{dom } f}$ for every $f \in \text{dom}(m)$. Let a be

the amalgamation of $(a_f)_{f \in \text{dom}(m)}$. Then

$$\begin{aligned} m_{\overline{a}^{F,A}}(f) &= [\overline{a}^{F,A} \cdot f]_{\text{dom } f} \\ &= \left[\overline{F(f)(a)}^{F, \text{dom } f} \right]_{\text{dom } f} \\ &= [\overline{a}_f^{F, \text{dom } f}]_{\text{dom } f} \\ &= [\dot{x}_f]_{\text{dom } f} \\ &= m(f) \end{aligned}$$

for every $f \in \text{dom}(m)$. Therefore, $\sigma_{F,A}(a) = [m_{\overline{a}^{F,A}}]_{K(F),A} = [m]_{K(F),A}$, which proves the surjectivity of $\sigma_{F,A}$.

For (b):

$$\begin{aligned} (\sigma_{F,A} \circ F(f))(b) &= \left[m_{\overline{F(f)(b)}^{F,A}} \right]_{K(F),A} \\ &= \left[m_{\overline{b}^{F,B} \cdot f} \right]_{K(F),A} \\ &= \left[m_{\overline{b}^{F,B} \cdot f} \right]_{K(F),A} \\ &= L_{K(F)}(f) \left(\left[m_{\overline{b}^{F,B}} \right]_{K(F),B} \right) \\ &= (L_{K(F)}(f) \circ \sigma_{F,B})(b) \end{aligned}$$

for every $b \in F(B)$.

For (c): Fix $a \in F(A)$. Now we claim that

$$\text{app}_A(K(\rho), m_{\overline{a}^{F,A}})(f) = m_{\overline{\rho_A(a)}^{G,A}}(f)$$

for every $f \in \text{dom}(\text{app}_A(K(\rho), m_{\overline{a}^{F,A}}))$. Indeed, $\text{app}_A(K(\rho), m_{\overline{a}^{F,A}})(f)$ is the $[\dot{y}]_{\text{dom } f}$ for which $\text{dom } f \Vdash_{(\mathcal{C}, J)} \dot{K}_{\rho, \text{dom } f}(\dot{x}) = \dot{y}$ for $[\dot{x}]_{\text{dom } f} := m_{\overline{a}^{F,A}}(f) = [\overline{a}^{F,A} \cdot f]_{\text{dom } f} = \left[\overline{F(f)(a)}^{F, \text{dom } f} \right]_{\text{dom } f}$. Then

$$\begin{aligned} \text{dom } f \Vdash_{(\mathcal{C}, J)} \dot{y} &= \dot{K}_{\rho, \text{dom } f}(\dot{x}) \\ &= \dot{K}_{\rho, \text{dom } f} \left(\overline{F(f)(a)}^{F, \text{dom } f} \right) \\ &= \overline{\rho_{\text{dom } f}(F(f)(a))}^{G, \text{dom } f} \\ &= \overline{G(f)(\rho_A(a))}^{G, \text{dom } f} \\ &= \overline{\rho_A(a)}^{G,A} \cdot f. \end{aligned}$$

So $\text{app}_A(K(\rho), m_{\overline{a}^{F,A}})(f) = [\dot{y}]_{\text{dom } f} = \left[\overline{\rho_A(a)}^{G,A} \cdot f \right]_{\text{dom } f} = m_{\overline{\rho_A(a)}^{G,A}}(f)$ as claimed. Therefore,

$$\begin{aligned} L_{K(\rho),A} \circ \sigma_{F,A}(a) &= L_{K(\rho),A} ([m_{\overline{a}^{F,A}}]_{K(F),A}) \\ &= [\text{app}_A(K(\rho), m_{\overline{a}^{F,A}})]_{K(G),A} \\ &= \left[m_{\overline{\rho_A(a)}^{G,A}} \right]_{K(G),A} \\ &= \sigma_{G,A} \circ \rho_A(a). \end{aligned}$$

□

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